Chapter 17

Geometric Thinking and Geometric Concepts

The geometry curriculum in grades K–8 should provide an opportunity to experience shapes in as many different forms as possible. These should include shapes built with blocks, sticks, or tiles; shapes drawn on paper or with a computer; and shapes observed in art, nature, and architecture. Hands-on, reflective, and interactive experiences are at the heart of good geometry activities at the elementary and middle school levels. The geometry curriculum should aim at the development of geometric reasoning and spatial sense. The three Big Ideas parallel three levels of thinking that characterize development over the K–8 school years.

BIG Ideas

1. Shapes, both two- and three-dimensional, exist in great variety. There are many different ways to see and describe similarities and differences among shapes. The more ways that one can classify and discriminate shapes, the better one understands them.
2. Shapes have properties that can be used when describing and analyzing them. Awareness of these properties helps us appreciate shapes in our world. Properties can be explored and analyzed in a variety of ways.
3. An analysis of geometric properties leads to deductive reasoning in a geometric environment.

THREE EXPLORATORY ACTIVITIES

To provide some common view of the nature of elementary and middle school geometry and how young children approach geometric concepts, three simple activities are offered here for you to do. The activities will provide some idea of the spirit of informal geometry as well as background for a discussion of children's geometric thinking. All you will need is a pencil, several pieces of paper, scissors, and 15 to 20 minutes.

DIFFERENT TRIANGLES

Draw a series of at least five triangles. After the first triangle, each new one should be different in some way from those already drawn. Write down why you think each is different.

SHAPES WITH TRIANGLES

Make a few copies of the 2-cm isometric grid found in the Blackline Masters, or simply place a sheet of paper over the grid. Draw three or four different figures by following the grid lines. Make each figure so that it has an area of 10 triangles. Count to find the distance around each figure (the perimeter), and record this next to each drawing. Examine your results for any ideas you may observe. Explore any ideas you have by drawing additional figures.

A TILING PATTERN

First make at least eight copies of any shape in Figure 17.1. An easy way to do this is to fold a piece of paper so that there are eight thicknesses. Trace the shape on an outside section, and cut through all eight thicknesses at once.

Think of the shapes you cut out as tiles. The task is to use the tiles to make a regular tiling pattern. A tiling pattern made with one shape has two basic properties. First, there are no holes or gaps. The tiles must fit together without overlapping and without leaving any spaces. Second, the tiles must be arranged in a repeating pattern that could be extended indefinitely. That is, if you were to tile an endless floor with your pattern, the design in one section of the floor would be the same as that in any other section. Several different tiling patterns are possible for each of the three tiles. Experiment to decide on a pattern that you like.

Notice that each of the tile shapes is made up of triangles and can be drawn on a triangle grid such as the 2-cm isometric grid or on the isometric dot grid, both found in the Blackline Masters. When you have decided on a tiling pattern, place a piece of paper over one of these two grids, and draw your tiling pattern using the grid as a guide. Cover most of the grid with your pattern.

Finally, suppose that your tiles come in two colors. With a pen or pencil, shade in some of the tiles to make a regular pattern in two colors.

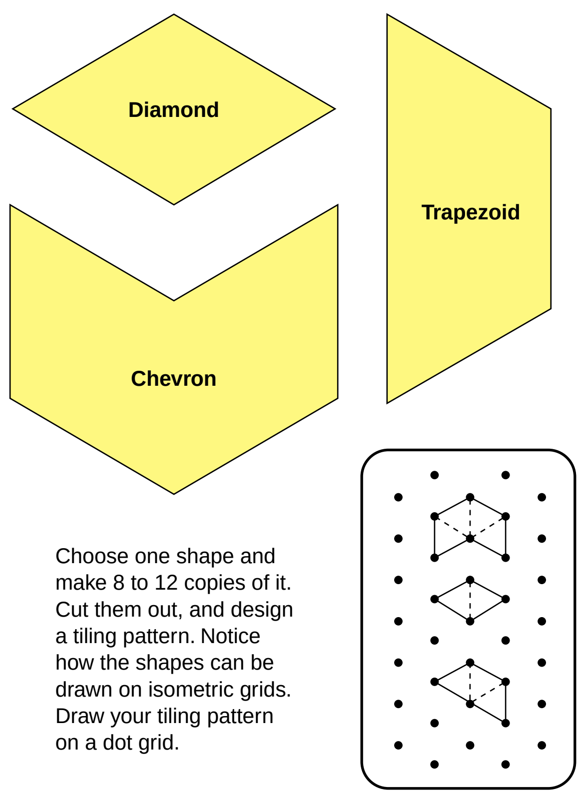

Figure 17.1 Three tile patterns.

Reflections on the Activities

The following observations apply to all geometry activities in school as well as the activities you have just completed.

Different People Think About Geometric Ideas in Different Ways

Compare your response to the three exploration activities with those of your peers. Are there qualitative differences as well as objective differences? How would primary-age children's approaches to these activities compare to an eighth grader's? Figure 17.2 (p. 308) shows how two students, one in the fifth grade and one in the eighth grade, responded to the triangle task. Research indicates that age is not the major criterion for how students think geometrically. The kinds of experiences a child has may be a more significant factor.

Explorations Can Help Develop Relationships

The more you play around with and think about the ideas in these activities, the more there is to think about. You might be able to extend each of these activities to develop the ideas beyond the obvious. For example:

FOR "DIFFERENT TRIANGLES"

How many different ways can two triangles be different? Could you draw five or more *quadrilaterals* that were each different?

FOR "SHAPES WITH TRIANGLES"

What did you notice about the shapes that had smaller perimeters as opposed to those with the larger perimeters? If you tried the same activity with rectangles on a square grid, what would the shapes with the largest and smallest perimeters look like? What about three-dimensional boxes? If you were to build different boxes with the same number of cubes, what could you say about the surface areas?

FOR "A TILING PATTERN"

How many different tiling patterns are there for this shape? Can any shape be used to tile with? Can you see any larger shapes within your pattern?

Notice that it takes more than just doing an activity to learn or create a new idea. The greatest learning occurs when you stop and reflect on what you did and begin to ask questions or make observations. Like all mathematics, geometry is best developed in a spirit of problem solving.

Geometry Activities and Hands-On Materials

Even the simple paper tiles used in "A Tiling Pattern" gave you the opportunity to explore spatial relationships and search for

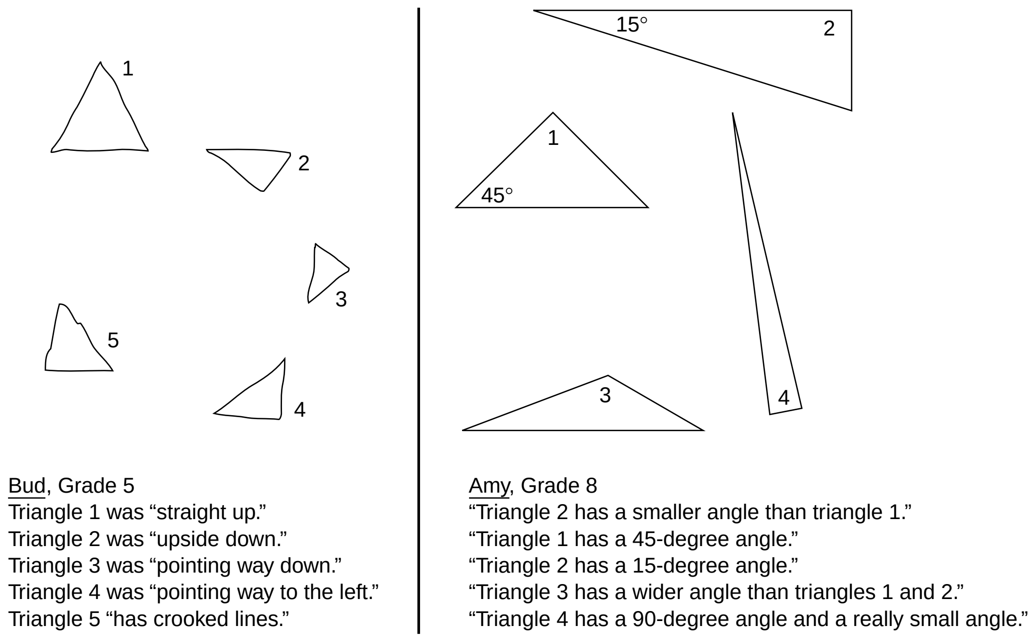

Figure 17.2 **Two children show markedly different responses to the task of drawing a series of different triangles.**

Source: From "Characterization of the van Hiele Levels of Development in Geometry," by W. F. Burger and J. M. Shaughnessy, 1986, *Journal for Research in Mathematics Education, 17*(1), pp. 38–39. Reprinted by permission of the National Council of Teachers of Mathematics.

patterns much more easily than without them. Activities on paper such as the dot grid in "Shapes with Triangles" are a second-best alternative to real physical objects. The same area and perimeter activity is much more effective with a collection of cardboard triangles that can be rearranged to form different shapes. The first activity is the least enticing of the three, but at least you could freely draw pictures. Virtually *every activity* that is appropriate for K–8 geometry should involve some form of hands-on materials, models, or at least paper (graph paper or dot paper) that lends itself to easy spatial explorations.

INFORMAL GEOMETRY AND SPATIAL SENSE

In *Principles and Standards for School Mathematics,* the authors chose only five broad content standards, one of which is Geometry. The prominence of this strand, appropriately kept separate from measurement, speaks to the importance that should be placed on the development of geometric ideas at all grade levels. Prior to the 1989 *Standards* document, geometry received only limited attention in the traditional curriculum. State standards now place a heavier emphasis on geometry than in the past. There is still a serious concern that in international comparisons, U.S. students fall short in this area.

Informal Geometry

The term *informal geometry* has been used for many years to refer to geometric activities appropriate for students in elementary and middle school. Informal geometry activities provide children with the opportunity to explore, to feel and see, to build and take apart, to make observations about shape in the world around them as well as in the world they create with drawings, models, and computers. Activities involve constructing, visualizing, comparing, transforming, and classifying geometric figures. The experiences and explorations can take place at different levels of sophistication: from shapes and their appearances to properties of shapes to relationships among properties. The spirit of informal geometry is one of exploration, almost always in a hands-on, engaging activity.

Spatial Sense

Just as a good definition of number sense is an intuition about numbers and their relationships, *spatial sense* can be defined as an intuition about shapes and the relationships among shapes. Individuals with spatial sense have a feel for the geometric aspects of their surroundings and the shapes formed by objects in the environment.

Many people say they aren't very good with shape or that they have poor spatial sense. The typical belief is that you are either born with spatial sense or not. This simply is not true! We now know that rich experiences with shape and spatial relationships, when provided consistently over time, can and do develop spatial sense. Without geometric experiences, most people do not grow in their spatial sense or spatial reasoning. Between 1990 and 1992, NAEP data indicated a significant improvement in students' geometric reasoning at all three grades tested, 4, 8, and 12 (Strutchens & Blume, 1997). Students did not just get smarter. What is more likely is that there has been an increasing emphasis on geometry at all grades. Still, much more needs to be done if U.S. children are to rise to the same level as their European and Asian counterparts.

The Importance of Geometry

In the past, most elementary and middle grades teachers spent very little time on geometry. Possibly they felt uncomfortable with the topic themselves or did not regard the topic as important. Traditional norm-referenced tests did not give a lot of weight to geometric thinking. Thanks to the increased NCTM emphasis on geometry and its inclusion in state testing programs, more geometry is being taught. Still, it is fair to ask, "Why study geometry?" Here are a few reasons that come to mind.

1. Geometry can provide a more complete appreciation of the world. Geometry can be found in the structure of the solar system, in geological formations, in rocks and crystals, in plants and flowers, even in animals. It is also a major part of our synthetic universe: Art, architecture, cars, machines, and virtually everything that humans create have elements of geometric form.
2. Geometric explorations can develop problem-solving skills. Spatial reasoning is an important form of problem solving, and problem solving is one of the major reasons for studying mathematics.
3. Geometry plays a key role in the study of other areas of mathematics. For example, fraction concepts are related to geometric part-to-whole constructs. Ratio and proportion are directly related to the geometric concept of similarity. Measurement and geometry are clearly related.
4. Geometry is used daily by many people. Scientists of all sorts, architects and artists, engineers, and land developers are just a few of the professions that use geometry regularly. At home, geometry helps build a fence, design a dog house, plan a garden, arrange a living room.
5. Geometry is enjoyable. If geometry increases students' fondness for mathematics more in general, that makes the effort worthwhile.

THE DEVELOPMENT OF GEOMETRIC THINKING

Until recently, the geometry curriculum in the United States has been poorly defined. Teachers and curriculum developers have had little guidance on what is important. However, the work of two Dutch educators, Pierre van Hiele and Dina van Hiele-Geldof, is beginning to have an impact on the design of geometry instruction and curriculum.

The van Hiele Levels of Geometric Thought

The van Hieles' work began in 1959 and immediately attracted a lot of attention in the Soviet Union but for nearly two decades got little notice in this country (Hoffer, 1983; Hoffer & Hoffer, 1992). But today, the van Hiele theory has become the most influential factor in the American geometry curriculum.

The most prominent feature of the model is a five-level hierarchy of ways of understanding spatial ideas. Each of the five levels describes the thinking processes used in geometric contexts. The levels describe how one thinks and what types of geometric ideas one thinks about, rather than how much knowledge one has. As one progresses from one level to the next, the object of one's geometric thinking changes.

Level 0: Visualization

The objects of thought at level 0 are shapes and what they "look like."

Students recognize and name figures based on the global, visual characteristics of the figure—a gestaltlike approach to shape. Students operating at this level are able to make measurements and even talk about properties of shapes, but these properties are not thought about explicitly. It is the appearance of the shape that defines it for the student. A square is a square "because it looks like a square." Because appearance is dominant at this level, appearances can overpower properties of a shape. For example, a square that has been rotated so that all sides are at a 45° angle to the vertical may not appear to be a square for a level 0 thinker. Students at this level will sort and classify shapes based on their appearances—"I put these together because they all look sort of alike."

The products of thought at level 0 are classes or groupings of shapes that seem to be "alike."

Level 1: Analysis

The objects of thought at level 1 are classes of shapes rather than individual shapes.

Students at the analysis level are able to consider all shapes within a class rather than a single shape. Instead of talking about *this* rectangle, it is possible to talk about *all* rectangles. By focusing on a class of shapes, students are able to think about what makes a rectangle a rectangle (four sides, opposite sides parallel, opposite sides same length, four right angles, congruent diagonals, etc.). The irrelevant features (e.g., size or orientation) fade into the background. At this level, students begin to appreciate that a collection of shapes goes together because of properties. Ideas about an individual shape can now be generalized to all shapes that fit that class. If a shape belongs to a particular class such as cubes, it has the corresponding properties of that class. "All cubes have six congruent faces, and each of those faces is a square." These properties were only implicit at level 0. Students operating at level 1 may be able to list all the properties of squares, rectangles, and parallelograms but not see that these are subclasses of one another, that all squares are rectangles and all rectangles are parallelograms. In defining a shape, level 1 thinkers are likely to list as many properties of a shape as they know.

The products of thought at level 1 are the properties of shapes.

Level 2: Informal Deduction

The objects of thought at level 2 are the properties of shapes.

As students begin to be able to think about properties of geometric objects without the constraints of a particular object, they are able to develop relationships between and among these properties. "If all four angles are right angles, the shape must be a rectangle. If it is a square, all angles are right angles. If it is a square, it must be a rectangle." With greater ability to engage in "if-then" reasoning, shapes can be classified using only minimum characteristics. For example, four congruent sides and at least one right angle can be sufficient to define a square. Rectangles are parallelograms with a right angle. Observations go beyond properties themselves and begin to focus on logical arguments *about* the properties. Students at level 2 will be able to follow and appreciate an informal deductive argument about shapes and their properties. "Proofs" may be more intuitive than rigorously deductive. However, there is an appreciation that a logical argument is compelling. An appreciation of the axiomatic structure of a formal deductive system, however, remains under the surface.

The products of thought at level 2 are relationships among properties of geometric objects.

Level 3: Deduction

The objects of thought at level 3 are relationships among properties of geometric objects.

At level 3, students are able to examine more than just the properties of shapes. Their earlier thinking has produced conjectures concerning relationships among properties. Are these conjectures correct? Are they "true"? As this analysis of the informal arguments takes place, the structure of a system complete with axioms, definitions, theorems, corollaries, and postulates begins to develop and can be appreciated as the necessary means of establishing geometric truth. At this level, students begin to appreciate the need for a system of logic that rests on a minimum set of assumptions and from which other truths can be derived. The student at this level is able to work with abstract statements about geometric properties and make conclusions based more on logic than intuition. This is the level of the traditional high school geometry course. A student operating at level 3 can clearly observe that the diagonals of a rectangle bisect each other, just as a student at a lower level of thought can. However, at level 3, there is an appreciation of the need to prove this from a series of deductive arguments. The level 2 thinker, by contrast, follows the argument but fails to appreciate the need.

The products of thought at level 3 are deductive axiomatic systems for geometry.

Level 4: Rigor

The objects of thought at level 4 are deductive axiomatic systems for geometry.

At the highest level of the van Hiele hierarchy, the object of attention is axiomatic systems themselves, not just the deductions within a system. There is an appreciation of the distinctions and relationships between different axiomatic systems. This is generally the level of a college mathematics major who is studying geometry as a branch of mathematical science.

The products of thought at level 4 are comparisons and contrasts among different axiomatic systems of geometry.

Characteristics of the van Hiele Levels

You no doubt noticed that the products of thought at each level are the same as the objects of thought at the next. This object-product relationship between levels of the van Hiele theory is illustrated in Figure 17.3. The objects (ideas) must be created at one level so that relationships among these objects can become the focus of the next level. In addition to this key concept of the theory, four related characteristics of the levels of thought merit special attention.

1. The levels are sequential. To arrive at any level above level 0, students must move through all prior levels. To move through a level means that one has experienced geometric thinking appropriate for that level and has created in one's own mind the types of objects or relationships that are the focus of thought at the next level. Skipping a level rarely occurs.
2. The levels are not age-dependent in the sense of the developmental stages of Piaget. A third grader or a high school student could be at level 0. Indeed, some students and adults remain forever at level 0, and a significant number of adults never reach level 2. But age is certainly related to the amount and types of geometric experiences that we have. Therefore, it is reasonable for all children in the K–2 range to be at level 0, as well as the majority of children in grades 3 and 4.
3. Geometric experience is the greatest single factor influencing advancement through the levels. Activities that permit children to explore, talk about, and interact with content at the next level, while increasing their experiences at their current level, have the best chance of advancing the level of thought for those children.
4. When instruction or language is at a level higher than that of the student, there will be a lack of communication. Students required to wrestle with objects of thought

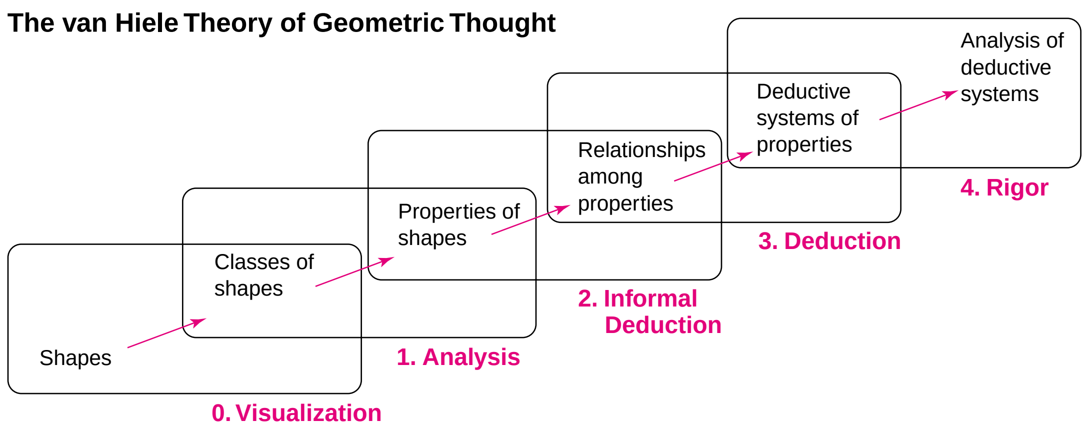

Figure 17.3 At each level of geometric thought, the ideas created become the focus or object of thought at the next level.

that have not been constructed at the earlier level may be forced into rote learning and achieve only temporary and superficial success. A student can, for example, memorize that all squares are rectangles without having constructed that relationship. A student may memorize a geometric proof but fail to create the steps or understand the rationale involved (Fuys, Geddes, & Tischler, 1988; Geddes & Fortunato, 1993).

Implications for Instruction

The van Hiele theory provides the thoughtful teacher with a framework within which to conduct geometric activities. The theory does not specify content or curriculum but can be applied to most activities. Most activities can be designed to begin with the assumption of a particular level and then be raised or lowered by means of the types of questioning and guidance provided by the teacher.

Instructional Goals: Content and Levels of Thought

The Geometry standard in *Principles and Standards* focuses on process as well as content. Verbs such as *describe, compare, relate, represent, investigate, sort, reason, analyze, predict, test* (conjectures), and *critique* (arguments) are all used in the description of the standard (see Appendix A). The goals are much broader than a collection of facts and bits of knowledge about geometric ideas. The term *spatial sense* best sums up the Geometry standard.

The van Hiele theory fits very nicely with a *Principles and Standards* view of geometry. It focuses our attention on how students think in geometric contexts and the object of their thinking: shapes ⟶ properties ⟶ informal logic ⟶ deductive principles. If the van Hiele theory is correct—and there is much evidence to support it—then a major goal of the K–8 curriculum must be to advance students' level of geometric thought. If students are to be adequately prepared for the deductive geometry curriculum of high school, their thinking should have advanced to at least level 2.

This is not to say that content knowledge is not important. Spatial sense is clearly enhanced by an understanding of shapes, what they look like, and even what they are named. The concepts of symmetry, congruence, and similarity contribute to understanding our geometric world. And the interaction with measurement that allows us to analyze angle measures and relationships between geometric entities is also valuable. But these must all be developed not in the context of "things to master" but rather as ways of knowing and understanding the geometric world.

Teaching at the Student's Level of Thought

A developmental approach to instruction demands that we listen to children and begin where we find them. The van Hiele theory highlights the necessity of teaching at the child's level. However, almost any activity can be modified to span two levels of thinking, even within the same classroom. We can respect the responses and observations made by children that suggest a lower level of thought while encouraging and challenging children to operate at the next level. Remember that it is the type of thinking that children are required to do that makes a difference in learning, not the specific content.

The following are some suggested features of instruction appropriate for the first three van Hiele levels.

<u>Features of Level 0 Activities</u>

- Involve lots of sorting, identifying, and describing of various shapes.
- Use lots of physical models that can be manipulated by the students.
- Include many different and varied examples of shapes so that irrelevant features do not become important. (Many students, for example, believe that only equilateral triangles are really triangles or that squares turned 45° are no longer squares.)

Provide opportunities to build, make, draw, put together, and take apart shapes.

Features of Level 1 Activities

Begin to focus more on properties of figures than on simple identification. Define, measure, observe, and change properties with the use of models.

Use problem-solving contexts in which properties of shapes are important components.

Continue to use models, as with level 0, but include models that permit the exploration of various properties of figures.

Classify figures based on properties of shapes as well as by names of shapes. For example, find different properties of triangles that make some alike and others different.

Features of Level 2 Activities

Continue to use models, with a focus on defining properties. Make property lists, and discuss which properties are necessary and which are sufficient conditions for a specific shape or concept.

Include language of an informal deductive nature: *all, some, none, if-then, what if,* and the like.

Investigate the converse of certain relationships for validity. For example, the converse of "If it is a square, it must have four right angles" is "If it has four right angles, it must be a square."

Use models and drawings as tools to think with, and begin to look for generalizations and counterexamples.

Encourage the making and testing of hypotheses.

Most of the content of the K–8 curriculum can be adapted to any of the three levels. An exception may be an inappropriate attention to abstract concepts such as point, line, ray, and plane as basic elements of geometric forms. These abstract ideas are not appropriate even at level 2.

Listen to your students during a geometry activity. Compare their comments and observations with the descriptions of the first two van Hiele levels. Be sure that the activities you plan do not require that students reason above their level of thought.

The activities suggested in the remainder of this chapter are grouped according to the first three van Hiele levels. These are just suggestions for getting started. Most activities have the potential of being addressed at slightly lower or higher levels.

The intent here is to illustrate the wide variety of things that can be done. Find ideas that you like, and develop them fully. Search out additional resource books to help you.

Van de Walle, John A. (2001). Geometric Thinking and Geometric Concepts. In *Elementary and Middle School Mathematics: Teaching Developmentally, 4th ed*. Boston: Allyn and Bacon.