Two Ways of Thinking about Division

By Donna Curry

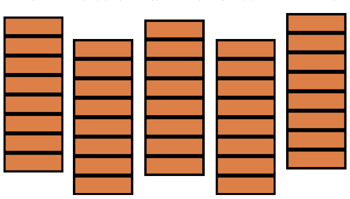
We know that division is the inverse of multiplication. We also know that the answer to a division problem can sometimes be found by repeated subtraction e.g., the answer to 18 ÷ 3, determined by seeing how many 3's can be subtracted from 18.

Many of us, though, have not been taught that there are two ways to look at division: **partitive** and **quotitive**. Understanding these two different models of division may help your students visualize division of fractions. Having students understand that there are two ways to look at division does ***not*** mean that students need to be taught the terminology, but rather that they understand that division can be represented by different models.

The Partitive Model. In this model, the idea behind division starts with knowing how many equal parts there are, and finding out the size of eachpart. The problem: ***If we have $20 to be shared among four people, howmuch does each person get?*** is an example of division as partitive. We usually intuitively develop this model first. [Someone using this model may pass out $1 to each person, continuing until there is nothing left – or at least not enough for everyone to have yet another dollar.]

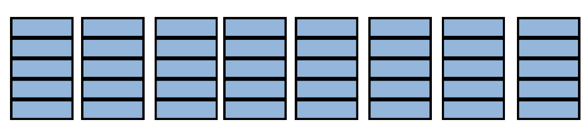
The Quotitive Model. This is also called a measurement model for division. Here, we start out knowing the size of the parts, but are asking how many of them exist. The problem: ***How many four-dollar tickets can you buy with $20?*** is an example of the quotitive model. The repeated subtraction strategy can be readily applied in this model since you know exactly how many to take for each group. In this example, you know to take groups of $4 away until there is nothing left (or at least not enough to make another group of 4). [Someone using the model may start with 20 dollar bills and place them in stacks of four until he runs out of bills.]

Why does it matter that there are two different ways of thinking about division? How we reason to figure out the answer differs with the two models. Think about this example: 40 ÷ 5.



In the partitive model, we might ask ourselves, "How many are in each of the five groups?" Visually we might see the picture on the left.

In the quotitive model, we ask ourselves, "How many groups are there if we have five in each group?" Visually, we have a different picture, as seen below.

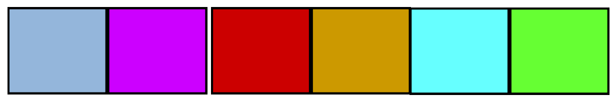


The same answer in both cases, yes? Well, yes and no. The result of the calculation is 8 in each situation. However, in the first situation we have not 8, but 8 per group. In the second situation, we have 8 different groups. Knowing whether the answer is 8 in a group or 8 groups is important, especially when it comes to dividing fractions.

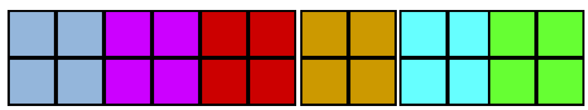
Students can reason about these ideas much more readily when there is meaning attached. 40 ÷ 5 has no meaning. "How many cases do I need if I can pack 5 disks in each case and I have a total of 40 disks?" gives students a sense that things are going to be grouped into sets of 5. Compare that situation to this one: "If I have 40 disks, how many disks can I fit into each of 5 cases so that there is an equal number in each case?"

As a teacher, if you know that there are two ways to pose a question about division, you can help students learn to do likewise. With division of whole numbers, both questions seem to be helpful for students to reason about the situation. In division of fractions, the quotitive model seems to be most effective in adding meaning.

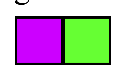
Let's progress from working with whole numbers to working with fractions (modeling a process you could use with your own students). First, think about the quotitive model for this situation: 6 ÷ 1. [How many 1-pound packages of hamburger could we make from a 6-pound package?] How many groups of 1 are there? Obviously, there are 6 groups of 1; hence, 6 ÷ 1 = 6.



What about 6 ÷ ¼? [How many 1/4-pound packages of hamburger could we make from a 6-pound package?] First, adding context gives the problem meaning. We are now going to put less hamburger than the previous example into each package so it makes sense that we should have more packages, not fewer than the previous example of 6 ÷ 1. Visually, we can see that we can get 4 ¼-pounds out of 1 pound. We can "see" why the problem becomes 6 x 4.

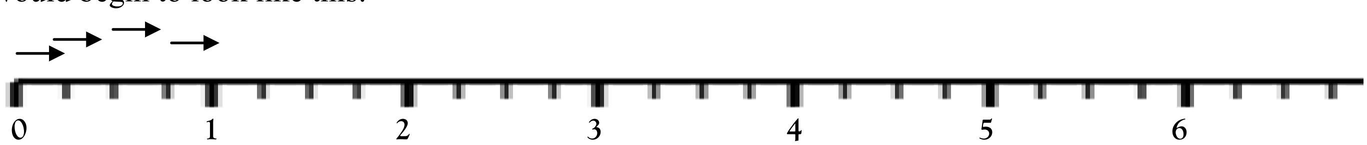


Let's look at this division problem: ½ ÷ 1/4. [How many 1/4-pound packages of hamburger could we make from a 1/2-pound package?] We use the same reasoning as we did with the examples above: "How many groups of ¼ are there in ½?" Since ¼ is less than ½, we know that we can get at least 1 package that has ¼-pound of hamburger in it. Visually, we see that we actually have two full ¼-pound packages.



Visually, you can see that there are two groups of ¼ in ½. The answer to ½ ÷ ¼ is not *just* 2, but two groups of ¼. Since dividing is supposed to give us a smaller answer (at least that is what many of our students have been told over the years), they may think that 2 cannot be a reasonable answer. Knowing that 2 represents two fourths makes much more sense.

Rather than shapes such as those above, you could also use the number line to illustrate these ideas. 6 ÷ 1/4 would begin to look like this:

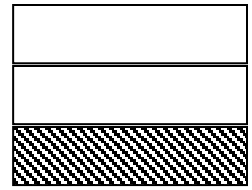


Even with the number line, it is clear that each whole unit is being divided into fourths, so there will be many fourths - more than the original number 6.

In case your students still do not quite believe you when you tell them that with division you sometimes get an answer larger than either number you started with, ask them to think about money (a context most of our students are very familiar with). Imagine this situation: I have a 50-cent piece. I need quarters for the vending machine. How many quarters can I get from ½-dollar? Or, in "naked numbers": ½ ÷ ¼ =?

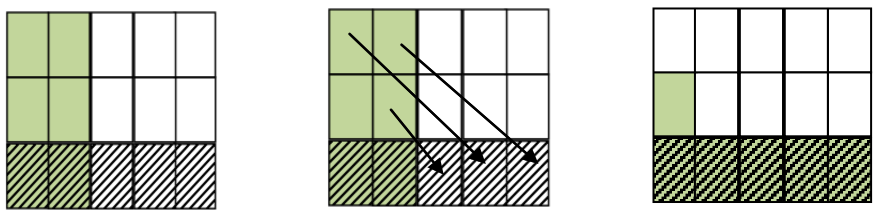
Moving back and forth between real-life situations and typical "naked number" problems, and thinking quotatively will help demystify division of fractions.

The examples above used a one-dimensional visual. Today, two-dimensional area models are sometimes used to explain how multiplication and division work. Let’s see how this thinking – and visualizing – works with dividing fractions less than 1. Think about 1/3 ÷ 2/5. First, reason to estimate an answer. You may think that 1/3 and 2/5 are somewhat similar in size, so could predict that the answer should be somewhat close to 1.



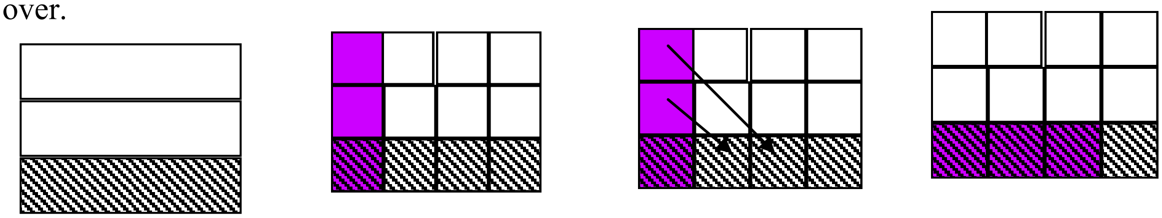
Let’s create an area model for division of 1/3 ÷ 2/5. First, create 1/3 (the shaded part on the right). Then use the same rectangle and divide it into 5 parts vertically. Shade in two parts to represent 2/5 of the whole as seen in the first picture below.

Move as many of the pieces of 2/5 into the 1/3 space as you can. There are 6 “pieces used to represent 2/5” (in the visual below those 6 pieces represent 6/15, which is the same as 2/5). There are only 5 pieces for 1/3 (5/15). You can see that that you can’t quite get a whole 2/5 out of 1/3. In fact, it appears that 5/6 of 2/5 fits into 1/3.



Let’s look at another example: 1/3 ÷ ¼. Ask, “How many ¼’s are there in 1/3?” [Before learning how to divide fractions, students should have already had extensive exposure to comparing fractions, so they should know that 1/3 is larger than ¼ - without having to figure it out by finding the greatest common denominator.]

Before drawing a picture, reason: since ¼ is less than 1/3, there should be at least one whole ¼ in 1/3 - with something left over.



One whole 1/4 fits into 1/3. So there is 1 fourth in 1/3. Since it takes 3 pieces (12ths) to make 1/4, we have 1 of those 3 left over. So 1/3 ÷ 1/4 = 1 1/3.

Providing opportunities for students to reason and visualize division will help them understand why dividing does not always yield a smaller number - that, especially dividing with fractions can produce answers larger than either of the two numbers.

Visualizing fractions might not work for you – after all, you were successful in math class, so you probably “got it” without having the have different models. However, many of our adult learners did not “get it” the first - or second – time they were asked to learn the various procedures, or algorithms, for dividing. For them, number lines and area models might be the key to opening up their understanding. You may want to practice these various models so you can effectively present them to your students. The National Library of Virtual Manipulatives website is one place where you (as well as your students) can “play”: http://nlvm.usu.edu/en/nav/frames_asid_193_g_2_t_1.html. If you prefer to deal with circles rather than rectangles, you might try this website: http://visualfractions.com/divide.htm. Another website that you might find useful for reviewing models of division with fractions is http://mysite.ncnetwork.net/res1359w3/mathmodels/div_mod.html. As you explore, you will find that there are various ways to visualize dividing fractions. Find one that works for you and then help your students learn how to create their own representations.

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